

ON THE CONVERGENCE OF LOCAL EXPANSIONS OF LAYER POTENTIALS

CHARLES L. EPSTEIN*, LESLIE GREENGARD†, AND ANDREAS KLÖCKNER†

Abstract. In this paper, we study the convergence behavior of local expansions of singular or weakly singular layer potentials defined on a smooth, closed boundary Γ , when evaluated on Γ itself. For suitably chosen centers, we derive estimates for the rate of convergence of the induced local expansions as the centers approach the boundary. These estimates provide the analytical foundation for a recently developed quadrature method called ‘quadrature by expansion’ (QBX). They may also be of mathematical interest, particularly for asymptotic analysis in potential theory.

Key words. Integral equations, layer potential, quadrature, singular integrals, spherical harmonics, expansion, Laplace equation, Helmholtz equation.

AMS subject classifications. 65R20, 65N38, 65N80, 31A10, 65D32

1. Introduction. The paper [16] describes a new method for the evaluation of layer potentials referred to as ‘quadrature by expansion.’ This method, denoted by QBX, is a technique for evaluating layer potentials *on surface*, which has simplified the development of fast, high-order accurate solvers for boundary integral equations. It is particularly easy to implement because it does not require the evaluation of integrals with non-smooth integrands. Here, we present the analytic foundations for QBX, which consist mainly in the establishment of decay estimates for one-sided local expansions induced by the layer potentials themselves. While introduced in the present context for the purpose of numerical analysis, these estimates may be of interest in their own right in asymptotic analysis and potential theory.

For simplicity we assume that $\Gamma \subset \mathbb{R}^n$ is a smooth compact hypersurface separating \mathbb{R}^n into two components $\mathbb{R}^n \setminus \Gamma = D_+ \cup D_-$. Throughout the paper, we let D_- denote the bounded component and D_+ the unbounded component. By a layer potential, we mean an integral of the form

$$F_{\pm}(x) = \int_{\Gamma} k(x, y) \varphi(y) dS(y), \text{ for } x \in D_{\pm}, \quad (1.1)$$

where $\varphi(y)$ is a smooth density on Γ and $k(x, y)$ is the Schwartz kernel, defined on $\mathbb{R}^n \times \mathbb{R}^n$, for a pseudodifferential operator, \mathcal{K} , which satisfies the transmission condition, see Section 18.2 in [14]. We are primarily interested in the case where $k(x, y)$ is the Green’s function for the Laplace or Helmholtz equation (or the Cauchy kernel), but more general cases also arise in a number of applications.

The kernels $k(x, y)$ in these calculations have singularities on the diagonal (when $x = y$) which have hampered the development of stable, efficient, and high order accurate methods for their evaluation, particularly on surfaces in \mathbb{R}^3 . The most frequently used high-order methods to date that are suitable for use with patch-based

* Depts. of Mathematics and Radiology, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104. E-mail: cle@math.upenn.edu. Research partially supported by NSF grants DMS09-35165 and DMS12-05851.

†Courant Institute, New York University, 251 Mercer Street, New York, NY 10012. E-mails: {greengard, kloeckner}@cims.nyu.edu. Research partially supported by the U.S. Department of Energy under contract DEFG0288ER25053 and by the Air Force Office of Scientific Research under NSSEFF Program Award FA9550-10-1-0180.

surface discretizations have largely been based on product integration. In this approach, the integral of the kernel multiplied by a piecewise polynomial approximation of the density φ on a piecewise smooth approximation of the boundary Γ is computed to high order using a mixture of analysis, linear algebraic techniques, and optimization. These have been developed in both two and three dimensions for a variety of kernels [6, 13, 25, 28]. Other methods, more closely related to QBX, are based on regularization of the kernel, combined with smooth quadrature rules and asymptotic correction [4, 10, 12, 19, 24]. Remarkably little use has been made, however, of the fact that the functions $F_{\pm}(x)$ have smooth extensions to $\overline{D_+}$ and $\overline{D_-}$. The limits from the two sides, of course, often differ.

REMARK 1. *An exception is [9, 15, 20], in which the authors do make use of a local expansion about an off-surface point. In those papers, however, the viewpoint is global. In essence, a single expansion center is introduced (for Cauchy integrals with analytic data), with a radius of convergence determined by the location of the nearest singularity of the analytic function itself.*

In our case, we consider boundary data of finite differentiability, which is not the restriction of a real- or complex-analytic function and hence make no assumptions about the location of the nearest underlying singularity. We are interested in the behavior of local expansions in balls centered at points close to the boundary Γ , with radius equal to the distance from the center to the boundary. Due to the nature of the kernel functions, $k(x, y)$, we can establish error estimates, valid in the closed ball, that only depend on the smoothness of the data and boundary near to the intersection of Γ with the boundary of the ball.

A more detailed review of existing approaches to quadrature is given in [16] and the textbooks [2, 5, 18].

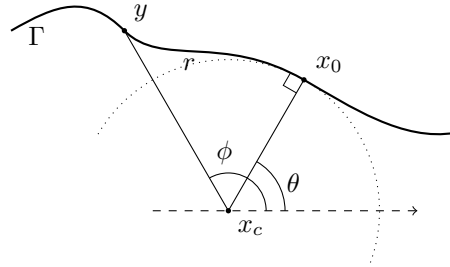


Fig. 1.1: A QBX expansion near a curve Γ , with a source point y , a target point x_0 , and an expansion center x_c . The angles ϕ and θ are mainly used in Section 2.3.

The QBX approach is based on the fact that, because of its one-sided smoothness, $F_{\pm}(x)$ supports a Taylor series expansion about x . More precisely, suppose that $x_0 \in \Gamma$, that $x_c \in D_-$ and that there is a ball of radius r about x_c , $B_r(x_c) \subset D_-$, such that $x_0 \in \partial B_r(x_c)$, where $\partial B_r(x_c)$ denotes the boundary of the ball. In other words, x_0 is a point of tangency of the ball and the surface Γ . (The analysis for the $+$ -case is essentially identical.) The fact that F_- is smooth in $\overline{B_r(x_c)}$ and the remainder theorem for Taylor series shows that

$$\left| F_-(x_0) - \sum_{\{\alpha: |\alpha| \leq k\}} \frac{\partial_x^\alpha F_-(x_c)}{\alpha!} (x_c - x_0)^\alpha \right| \leq M_k(F_-) r^{k+1}. \quad (1.2)$$

Here α denotes a standard multi-index in n dimensions, and $M_k(F_-)$ is a constant

depending on the \mathcal{C}^{k+1} -norm of F_- in $B_r(x_c)$. Because \mathcal{K} satisfies the transmission condition, this quantity is bounded by the Hölder $\mathcal{C}^{k+2+m,\alpha}$ -norm of φ , where the m is the order of \mathcal{K} , and α is any positive number. The two key observations underlying QBX are that the error in the expansion (1.2) can be controlled and that the derivatives $\{\partial_x^\alpha F_-(x_c)\}$ can be computed accurately using only standard quadrature techniques for *smooth* functions.

Briefly stated, QBX consists of a three-step procedure: For each boundary point x_0 , we

1. find a nearby off-surface point x_c ,
2. construct a local expansion of the form (1.2) about the center x_c of the field induced by the layer potential, and
3. evaluate a partial sum of the local expansion from the previous step at x_0 .

For kernels that satisfy a classical PDE, more efficient series expansion are preferable, based on standard separation of variables. For example, if \mathcal{K} is a single or double layer potential for the Helmholtz equation in three dimensions, for which the kernel is $k(x, y) = e^{ik|x-y|}/(4\pi|x-y|)$, then it is convenient to use a (doubly-indexed) spherical harmonic expansion instead of a (triply-indexed) Taylor series.

We restrict our attention in this paper to kernels $k(x, y)$ which are Green's functions of some constant-coefficient elliptic partial differential operator, which are well-known to have their singular support along the diagonal. The analysis below, therefore, applies to rectifiable hypersurfaces Γ , provided that Γ is smooth (\mathcal{C}^∞) in a neighborhood of the point, x , where we are trying to estimate the value of $F_\pm(x)$. To simplify the discussion below we assume that Γ is globally infinitely differentiable. This is not necessary; from our analysis it is clear that the accuracy of this approach near a point x_0 on the boundary depends only on the smoothness of the boundary and the data near to x_0 .

The purpose of this paper is to provide rigorous error estimates for the QBX procedure outlined above. In Sections 2 and 3, we derive decay estimates for analogues of $M_k(F_-)$ in (1.2) for a selection of commonly encountered kernels. In Section 4, we couple this analysis with quadrature error estimates for the numerical computation of $\{\partial_x^\alpha F_-(x_c)\}$. The corresponding analysis for the case of a general kernel is omitted. The steps involved are essentially the same as for the cases studied here.

2. The two-dimensional case. We first consider two dimensional examples arising in potential and scattering theory, namely the Cauchy kernel, the Green's function for the Laplace equation, and the Green's function for the Helmholtz equation.

2.1. The Cauchy Kernel. Let D_- be a bounded domain in $\mathbb{C} = \mathbb{R}^2$ with a smooth boundary $\Gamma = \partial D_-$, and φ a function on Γ . The Cauchy formula defines an analytic function f in $\mathbb{C} \setminus \Gamma$:

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta) d\zeta}{\zeta - z}. \quad (2.1)$$

We denote the restrictions $f_- = f|_{D_-}$ and $f_+ = f|_{D_+}$ and recall the well-known jump formula:

$$[f_+ - f_-]|_{\Gamma} = \varphi. \quad (2.2)$$

The crucial fact in our analysis is that the functions f_{\pm} extend to the closures of their domains of definition to be essentially as smooth as φ .

REMARK 2. *If smoothness is measured in terms of L^2 -Sobolev norms, then f_{\pm} have a half-derivative more than, φ , whereas, if we use standard Hölder norms, then f_{\pm} have the same regularity, up to the boundary as the boundary data. Of course f_{\pm} are analytic functions in the interior, so here we are primarily speaking of the boundary regularity.*

For purposes of definiteness, we suppose that $z_c \in D_-$, that the disk $B_r(z_c) \subset D_-$, and that $z_0 \in bB_r(z_c) \cap \Gamma$, where $z_0 = z_c + re^{i\theta_0}$. In this disk:

$$f_-(z) = \sum_{j=0}^{\infty} \frac{f_-^{(j)}(z_c)}{j!} (z - z_c)^j, \quad (2.3)$$

where $f_-^{(j)}(z) = \partial_z^j f_-(z)$. In general this power series converges in $B_r(z_c)$, but in no larger disk. If φ has more than a half an L^2 -derivative, or is Hölder continuous, then the series representation for f_- converges uniformly on $\overline{B_r(z_c)}$.

In the QBX approach, we would like to approximate the integral f_- from (2.1), at the boundary point z_0 , by using a finite partial sum of the Taylor series (2.3), rather than a quadrature rule that is designed to handle the singular kernel head-on. Because of the special structure of the Cauchy kernel, two different approaches are available to estimate the error:

$$e_N(z) = f_-(z) - \sum_{j=0}^N \frac{f_-^{(j)}(0)}{j!} (z - z_c)^j, \quad (2.4)$$

for $z = z_c + re^{i\theta_0}$. The first method does not generalize, but gives a simple explanation as to why this approach works, whereas the second gives a clear path to an error estimate in the general case.

Let $w(t)$ be an arc-length parametrization for Γ . The Cauchy integral is then given by

$$f(z) = \frac{1}{2\pi i} \int_0^L \frac{\varphi(w(t))w'(t)}{w(t) - z} dt. \quad (2.5)$$

To simplify notation, let us assume (without loss of generality) that $z_c = 0$. For $z \in B_r(0)$, by considering the series for $(1 - z/w(t))^{-1}$, we obtain

$$e_N(z) = \frac{1}{2\pi i} \sum_{j=N+1}^{\infty} \int_0^L \varphi(w(t)) \left(\frac{z}{w(t)} \right)^j \frac{w'(t)}{w(t)} dt. \quad (2.6)$$

As

$$\frac{w'(t)}{(w(t))^{j+1}} = \frac{-1}{j} \partial_t \frac{1}{(w(t))^j}, \quad (2.7)$$

then we can integrate by parts to obtain

$$e_N(z) = \frac{z}{2\pi i} \sum_{j=N+1}^{\infty} \int_0^L \partial_t [\varphi(w(t))] \left(\frac{z}{w(t)} \right)^{j-1} \frac{dt}{j \cdot w(t)}. \quad (2.8)$$

In order to integrate by parts again, we use the fact that, for $j > 1$,

$$\frac{1}{[w(t)]^j} = \frac{-1}{(j-1)w'(t)} \partial_t \left(\frac{1}{[w(t)]^{j-1}} \right). \quad (2.9)$$

If we define the first order differential operator D_t by

$$D_t g = \partial_t \left(\frac{g(t)}{w'(t)} \right), \quad (2.10)$$

then we can integrate by parts N more times to obtain that

$$e_N(z) = \frac{z^{N+1}}{2\pi i} \sum_{j=N+1}^{\infty} \int_0^L D_t^N \partial_t [\varphi(w(t))] \underbrace{\left(\frac{z}{w(t)} \right)^{j-N-1}}_{|\cdot| \leq 1} \frac{(j-N-1)!}{w(t)j!} dt. \quad (2.11)$$

For $N \geq 1$, it is then straightforward to see that there is a constant, $M_{\Gamma, N}$, depending only on Γ and N such that

$$|e_N(z)| \leq M_{\Gamma, N} \|\varphi\|_{C^{N+1}(\Gamma)} |z|^{N+1} \quad (2.12)$$

for any z with $|z| \leq r$. This proves

THEOREM 2.1. *Let Γ be a smooth, bounded curve in \mathbb{C} such that $B_r(0) \subset \Gamma^c$, but $re^{i\theta_0} \in \Gamma \cap bB_r(0)$, where Γ^c denotes the complement of $\Gamma \subset \mathbb{R}^2$. For each positive integer N there is a constant $M_{\Gamma, N}$ such that, for $\varphi \in C^{N+1}(\Gamma)$, the error in the truncated Taylor series approximation is given by*

$$\left| \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(w)dw}{w - re^{i\theta_0}} - \sum_{j=0}^N \frac{f_-^{(j)}(0)}{j!} r^j e^{ij\theta_0} \right| \leq M_{\Gamma, N} \|\varphi\|_{C^{N+1}(\Gamma)} r^{N+1}. \quad (2.13)$$

Note that the coefficients of the expansion can be obtained by evaluating the non-singular integrals:

$$\frac{f_-^{(j)}(0)}{j!} = \frac{1}{2\pi i} \int_0^L \frac{\varphi(w(t))w'(t)dt}{[w(t)]^{j+1}}. \quad (2.14)$$

Error estimates for the numerical computation of these integrals are discussed in section 4.

We turn now to a second approach for estimating the error in truncating the Taylor series, which uses the Fourier expansion of f_- on $bB_r(0)$ directly. As f_- is holomorphic in $B_r(0)$ its Fourier series on the boundary only has terms with non-negative exponents:

$$f_-(re^{i\theta}) = \sum_{j=0}^{\infty} a_j e^{ij\theta}, \quad (2.15)$$

where

$$a_j = \frac{1}{2\pi} \int_0^{2\pi} f_-(re^{i\theta}) e^{-ij\theta} d\theta. \quad (2.16)$$

We again want to show that

$$\left| f_-(re^{i\theta}) - \sum_{j=0}^N a_j e^{ij\theta} \right| = O(r^{N+1}), \quad (2.17)$$

which is evidently a matter of estimating the Fourier coefficients themselves.

Integrating by parts, and assuming $j > 0$, we have

$$a_j = \frac{1}{2\pi} \left(\frac{r}{j} \right) \int_0^{2\pi} f_-^{(1)}(re^{i\theta}) e^{-i(j-1)\theta} d\theta. \quad (2.18)$$

Note that $e^{-ij\theta}$ has been replaced by $e^{-i(j-1)\theta}$, which is a consequence of the fact that

$$\partial_\theta f_-(re^{i\theta}) = ire^{i\theta} f_-^{(1)}(re^{i\theta}). \quad (2.19)$$

This shift in degree reappears when we consider spherical harmonic expansions below.

If f_- has $N+1$ derivatives on $bB_r(0)$, then we can repeat this $(N+1)$ -times to obtain that, for $j \geq N+1$,

$$a_j = \frac{1}{2\pi i} \left(\frac{r^{N+1}(j-N-1)!}{j!} \right) \int_0^{2\pi} f_-^{(N+1)}(re^{i\theta}) e^{-i(j-N-1)\theta} d\theta, \quad (2.20)$$

from which the estimate in (2.17) is immediate. The implied constant depends on $\|f_-\|_{C^{N+1}(B_r(0))}$, which we know from the mapping properties of the Cauchy integral operator is of the order $O(\|\varphi\|_{C^{N+1,\beta}(\Gamma)})$ for any $\beta > 0$, see Theorem 1.4 on page 147 of [27]. We therefore get an estimate slightly weaker than that obtained in (2.13).

2.2. Harmonic Layer Potentials. We now apply similar ideas to the evaluation of single and double layer potentials on the boundaries of planar regions. The fundamental solution for the Laplace operator is given by

$$G_0(z, w) = \frac{1}{2\pi} \log |z - w| = \frac{1}{2\pi} \operatorname{Re} \log(z - w). \quad (2.21)$$

While the complex log is multi-valued, its real part and its complex derivative are globally defined as single-valued analytic functions. As before we let D_- be a bounded region with boundary Γ . Assuming that φ is real valued, the single layer potential with density supported on Γ is defined as

$$u(\xi_1, \xi_2) = \frac{1}{2\pi} \operatorname{Re} \left[\int_0^L \varphi(w(t)) \log(w(t) - z) dt \right], \quad (2.22)$$

where $z = \xi_1 + i\xi_2$ and $w(t)$ is again an arc-length parametrization of Γ , viewed as lying in the complex plane.

We assume (without loss of generality) that the origin $0 \in D_-$, which we take to be the center of our power series expansion. If $B_r(0) \subset D_-$, then for $z \in B_r(0)$ we can rewrite this integral as

$$u(\xi_1, \xi_2) = \frac{1}{2\pi} \operatorname{Re} \left[\int_0^L \varphi(w(t)) \log \left(1 - \frac{z}{w(t)} \right) dt \right] + A_0, \quad (2.23)$$

where

$$A_0 = \frac{1}{2\pi} \int_0^L \varphi(w(t)) \log |w(t)| dt. \quad (2.24)$$

Let us denote by u_1 the first term in (2.23). As $|z/w(t)| < 1$, the power series expansion for the principal branch of $\log(1 - \zeta)$ about $\zeta = 0$, shows that u_1 has the following power series expansion:

$$u_1(\xi_1, \xi_2) = \frac{-1}{2\pi} \operatorname{Re} \left[\int_0^L \varphi(w(t)) \sum_{j=1}^{\infty} \frac{1}{j} \left(\frac{z}{w(t)} \right)^j dt \right]. \quad (2.25)$$

To approximate $u(\xi_1, \xi_2)$ for $z \in \overline{B_r(0)}$ we use A_0 plus a finite partial sum of this series, giving an error term:

$$\begin{aligned} e_N(\xi_1, \xi_2) = u(\xi_1, \xi_2) - \left[A_0 - \frac{1}{2\pi} \operatorname{Re} \left\{ \int_0^L \varphi(w(t)) \sum_{j=1}^N \frac{1}{j} \left(\frac{z}{w(t)} \right)^j dt \right\} \right] = \\ \frac{1}{2\pi} \sum_{j=N+1}^{\infty} \operatorname{Re} \left[\int_0^L \varphi(w(t)) \frac{1}{j} \left(\frac{z}{w(t)} \right)^j dt \right] \end{aligned} \quad (2.26)$$

Using the identity in equation (2.9) we can integrate by parts N -times to obtain:

$$e_N(\xi_1, \xi_2) = \frac{1}{2\pi} \sum_{j=N+1}^{\infty} \frac{z^{N+1}(j-N)!}{j!} \operatorname{Re} \left[\int_0^L D_t^N [\varphi(w(t))] \left(\frac{z}{w(t)} \right)^{j-N-1} \frac{dt}{w(t)} \right], \quad (2.27)$$

where D_t is again the differential operator defined in (2.10). As $r \rightarrow 0$,

$$\int_{\Gamma} \frac{dt}{|w(t)|} \propto -\log r. \quad (2.28)$$

The preceding estimates yield the following theorem.

THEOREM 2.2. *Suppose that Γ is a smooth, bounded curve embedded in \mathbb{R}^2 , such that $B_r(0) \subset \Gamma^c$, but $z_0 = re^{i\theta_0} \in bB_r(0) \cap \Gamma$. For N a positive integer, there is a constant $M_{N,\Gamma}$ so that if $\varphi \in \mathcal{C}^N(\Gamma)$, then*

$$\begin{aligned} \left| \int_{\Gamma} G_0(z_0, y) \varphi(y) ds(y) - \left[A_0 - \frac{1}{2\pi} \operatorname{Re} \left\{ \int_0^L \varphi(w(t)) \sum_{j=1}^N \frac{1}{j} \left(\frac{z}{w(t)} \right)^j dt \right\} \right] \right| \\ \leq M_{N,\Gamma} \|\varphi\|_{\mathcal{C}^N(\Gamma)} r^{N+1} \log \frac{1}{r}. \end{aligned} \quad (2.29)$$

The constant A_0 is defined in (2.24).

The double layer potential is defined as

$$v(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{\Gamma} \partial_{\mathbf{n}_w} \log |w - z| \varphi(w(t)) dt, \quad (2.30)$$

where \mathbf{n} is the outward normal along bD_- . As $\log(w - z)$ is analytic, the Cauchy-Riemann equations imply that

$$\partial_{\mathbf{n}_w} \log |w - z| = \operatorname{Re}[\partial_{\mathbf{n}_w} \log(w - z)] = -\operatorname{Im}[\partial_{\boldsymbol{\tau}_w} \log(w - z)], \quad (2.31)$$

where $\boldsymbol{\tau}$ is the unit tangent vector along TD_- . Thus, we may write

$$\partial_{\mathbf{n}_w} \log |w - z| = -\operatorname{Im} \frac{w'(t)}{w(t) - z}. \quad (2.32)$$

It follows that the estimate proved in the previous section for the Cauchy transform can be applied to analyze the error when replacing the double layer potential, $v(\xi_1^0, \xi_2^0)$ with the finite partial sum:

$$v_{(N)}(r, \theta_0) = -\frac{1}{2\pi} \sum_{j=0}^N \operatorname{Im} \left[\int_{\Gamma} \left(\frac{re^{i\theta_0}}{w(t)} \right)^j \frac{\varphi(w(t))w'(t)dt}{w(t)} \right] \quad (2.33)$$

where (r, θ_0) are the polar coordinates of (ξ_1^0, ξ_2^0) with respect to the origin (which is the center of the local expansion).

COROLLARY 2.3. *Let Γ be a smooth, bounded curve in \mathbb{C} such that $B_r(0) \subset \Gamma^c$, with $z_0 = \xi_1^0 + i\xi_2^0 = re^{i\theta_0} \in \Gamma \cap bB_r(0)$. For each positive integer N there is a constant $M_{N,\Gamma}$ such that if $\varphi \in C^{N+1}(\Gamma)$, then*

$$|v(\xi_1^0, \xi_2^0) - v_{(N)}(r, \theta_0)| \leq M_{N,\Gamma} r^{N+1} \|\varphi\|_{C^{N+1}(\Gamma)}, \quad (2.34)$$

where v is defined in (2.30) and $v_{(N)}$ is defined in (2.33).

2.3. Layer Potentials for the Helmholtz Equation. Our last example in two dimensions concerns layer potentials that satisfy the Helmholtz equation:

$$(\Delta + k^2)u = 0. \quad (2.35)$$

The fundamental solution, for $k \neq 0$, is given by the zeroth order Hankel function of the first kind:

$$G_k(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|). \quad (2.36)$$

It is well-known that the kernel $G_k(x, y)$ defines a classical pseudodifferential operator of order -2 , see [26]. For our purposes, the most important fact about this function is the Graf addition theorem, see equation 9.1.79 in [1], which gives the series representation:

$$H_0^{(1)}(k|x - y|) = \sum_{l=-\infty}^{\infty} J_{|l|}(k|x|) H_{|l|}^{(1)}(k|y|) e^{il(\theta - \phi)}, \quad (2.37)$$

valid so long as

$$|x| < |y| \text{ where } x = |x|e^{i\theta} \text{ and } y = |y|e^{i\phi}. \quad (2.38)$$

Here J_l and $H_l^{(1)}$ denote the usual Bessel and Hankel functions of the first kind, respectively.

We assume (without loss of generality) that the origin $0 \in D_-$, which we take to be the center of our series expansion. With these assumptions about D_- and $B_r(0)$, we define the single layer potential:

$$\begin{aligned} u(x) &= \int_{\Gamma} \varphi(y) G_k(x, y) ds(y) \\ &= \sum_{l=-\infty}^{\infty} J_{|l|}(k|x|) e^{il\theta} \int_{\Gamma} H_{|l|}^{(1)}(k|y|) e^{-il\phi_y} \varphi(y) ds(y) \\ &= \sum_{l=-\infty}^{\infty} \alpha_l J_{|l|}(k|x|) e^{il\theta}. \end{aligned} \quad (2.39)$$

The second equation follows from (2.37). The series in the last line is valid for $x \in B_r(0)$, with the coefficients $\{\alpha_l\}$ defined by the integrals:

$$\alpha_l = \int_{\Gamma} H_{|l|}^{(1)}(k|y|) e^{-il\phi_y} \varphi(y) ds(y). \quad (2.40)$$

We suppose that $\varphi \in \mathcal{C}^{N,\beta}(\Gamma)$, for some $\beta > 0$, so that u_{\pm} belongs to $\mathcal{C}^{N+1,\beta}(\overline{D_{\pm}})$. If we assume that $x_0 = re^{i\theta_0} \in bB_r(0) \cap \Gamma$, then, in the QBX approach, we would like to use the partial sum of the series:

$$\sum_{l=-N}^N \alpha_l J_{|l|}(kr) e^{il\theta_0} \quad (2.41)$$

as an approximation for $u_-(x_0)$. Because the Fourier series representation is unique, it follows that

$$\alpha_l J_{|l|}(kr) = \frac{1}{2\pi} \int_0^{2\pi} u_-(r \cos \theta, r \sin \theta) e^{-il\theta} d\theta. \quad (2.42)$$

We now seek, as above, to estimate the error

$$\left| u_-(x_0) - \sum_{l=-N}^N \alpha_l J_{|l|}(kr) e^{il\theta_0} \right|. \quad (2.43)$$

We make use of the integration by parts formula:

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} u_-(r \cos \theta, r \sin \theta) e^{-il\theta} d\theta &= \\ \frac{r}{2\pi \cdot l} \int_0^{2\pi} \left([\partial_{\bar{z}} u_-](r \cos \theta, r \sin \theta) e^{-i(l+1)\theta} - [\partial_z u_-](r \cos \theta, r \sin \theta) e^{-i(l-1)\theta} \right) d\theta, \end{aligned} \quad (2.44)$$

where

$$\partial_z = \frac{1}{2}(\partial_x + i\partial_y) \text{ and } \partial_{\bar{z}} = \frac{1}{2}(\partial_x - i\partial_y). \quad (2.45)$$

If we integrate by parts, in this manner, $N + 1$ times, then the order l term, where $N < |l|$, produces an integrand of the form:

$$r^{N+1} \sum_{j=0}^{N+1} c_{l,N+1,j} e^{i(l+2j-N-1)\theta} [\partial_z^j \partial_{\bar{z}}^{N+1-j} u_-](r \cos \theta, r \sin \theta). \quad (2.46)$$

Each coefficient $c_{l,N+1,j}$ is a sum of $\binom{N+1}{j}$ terms of the form:

$$\frac{1}{l(l + \epsilon_1) \cdots (l + \epsilon_N)} \text{ where } \epsilon_j \in \{-N, 1 - N, \dots, N - 1, N\}, \quad (2.47)$$

with each satisfying the upper bound:

$$|c_{l,N+1,j}| \leq \frac{(|l| - N - 1)!}{|l|!}. \quad (2.48)$$

Thus, for $|l| > N$, summing over j gives the estimate

$$|\alpha_l J_{|l|}(kr)| \leq \frac{(2r)^{N+1} (|l| - N - 1)! \cdot \|u_-\|_{\mathcal{C}^{N+1}(\overline{B_r(0)})}}{|l|!}. \quad (2.49)$$

Hence, for $N \geq 1$, there is a constant M_N so that

$$\left| u_-(re^{i\theta_0}) - \sum_{l=-N}^N \alpha_l J_{|l|}(kr) e^{il\theta_0} \right| \leq M_N r^{N+1} \|u_-\|_{\mathcal{C}^{N+1}(\overline{B_r(0)})}. \quad (2.50)$$

The operator G_k maps $\mathcal{C}^{N,\beta}(\Gamma)$ to $\mathcal{C}^{N+1,\beta}(\overline{D_-})$ for any $\beta > 0$, see [7]. Hence the preceding estimates yield the following result.

THEOREM 2.4. *Suppose that Γ is a smooth, bounded curve embedded in \mathbb{R}^2 , such that $B_r(0) \subset \Gamma^c$, with $re^{i\theta_0} \in bB_r(0) \cap \Gamma$. For $k \in [0, \infty)$, N a positive integer, and $\beta > 0$, there is a constant $M'_{N,\beta}$ so that if $\varphi \in \mathcal{C}^{N,\beta}(\Gamma)$, then*

$$\left| \int_{\Gamma} G_k(re^{i\theta_0}, y) \varphi(y) ds(y) - \sum_{l=-N}^N \alpha_l J_{|l|}(kr) e^{il\theta_0} \right| \leq M'_{N,\beta} r^{N+1} \|\varphi\|_{\mathcal{C}^{N,\beta}(\Gamma)}. \quad (2.51)$$

Here the coefficients $\{\alpha_l\}$ are given by (2.40).

An estimate for the double layer

$$v(x) = \int_{\Gamma} \partial_{\mathbf{n}_y} G_k(x, y) \varphi(y) ds(y) \quad (2.52)$$

is obtained just as easily. In D_- , the double layer has an expansion of the form

$$v_-(x) = \sum_{l=-\infty}^{\infty} \alpha_l J_{|l|}(k|x|) e^{il\theta}, \quad (2.53)$$

valid for $x \in B_r(0)$, where now the coefficients $\{\alpha_l\}$ are defined by the integrals:

$$\alpha_l = \int_{\Gamma} \partial_{\mathbf{n}_y} [H_{|l|}^{(1)}(k|y|)e^{-il\phi_y}] \varphi(y) ds(y). \quad (2.54)$$

Arguing exactly as before we can show that, for $N \geq 2$, we have the estimate:

$$\left| v(x_0) - \sum_{l=1-N}^{N-1} \alpha_l J_{|l|}(k|x|)e^{il\theta} \right| \leq M_N r^N \|v_-\|_{\mathcal{C}^N(\overline{B_r(0)})}. \quad (2.55)$$

Because the double layer defines an operator of order -1 we have

THEOREM 2.5. *Suppose that Γ is a smooth, bounded curve embedded in \mathbb{R}^2 , such that $B_r(0) \subset \Gamma^c$, but $re^{i\theta_0} \in \partial B_r(0) \cap \Gamma$. For $k \in [0, \infty)$, N a positive integer, and $\beta > 0$, there is a constant $M''_{N,\beta}$ so that if $\varphi \in \mathcal{C}^{N,\beta}(\Gamma)$, then*

$$\left| \int_{\Gamma} \partial_{\mathbf{n}_y} G_k(re^{i\theta_0}, y) \varphi(y) ds(y) - \sum_{l=1-N}^{N-1} \alpha_l J_{|l|}(kr) e^{il\theta_0} \right| \leq M''_{N,\beta} r^N \|\varphi\|_{\mathcal{C}^{N,\beta}(\Gamma)}. \quad (2.56)$$

Here the coefficients $\{\alpha_l\}$ are given by (2.54).

3. Three-dimensional Layer Potentials. In this section we consider the evaluation of layer potentials arising from the Helmholtz equation in three dimensions. The “outgoing” fundamental solution is given by

$$G_k(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|} \quad (3.1)$$

for $\text{Im } k \geq 0$. The relevant formulæ in the harmonic ($k = 0$) case are somewhat different, but follow the same lines as those for $k \neq 0$. We leave that analysis to the interested reader.

As in the two-dimensional case, this Green’s function also has a classical expansion in terms of spherical harmonics [21]:

$$G_k(x, y) = ik \sum_{l=0}^{\infty} j_l(k|x|) h_l(k|y|) \sum_{m=-l}^l Y_l^m(\theta_1, \phi_1) Y_l^{-m}(\theta_2, \phi_2) \quad (3.2)$$

assuming $|x| < |y|$, with

$$x = |x|\omega(\theta_1, \phi_1) \text{ and } y = |y|\omega(\theta_2, \phi_2), \quad (3.3)$$

where

$$\omega(\theta, \phi) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi). \quad (3.4)$$

Here, j_l and h_l denote the spherical Bessel and Hankel functions of the first kind, respectively [1]. The normalized spherical harmonics Y_l^m are defined for $|m| \leq l$ by

$$Y_l^m(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \cdot \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos \phi) e^{im\theta}, \quad (3.5)$$

where P_l^m is the associated Legendre function of degree l and order m .

We let D_- denote a bounded region in \mathbb{R}^3 with smooth boundary Γ . We assume, as in the two-dimensional case, that the origin $0 \in D_-$ and that $B_r(0) \subset D_-$, with $x_0 \in bB_r(0) \cap \Gamma$. If φ is an integrable function defined on Γ , then a single layer potential with this density is given by

$$\begin{aligned} f(x) &= \int_{\Gamma} G_k(x, y) \varphi(y) dS(y) \\ &= \sum_{l=0}^{\infty} j_l(k|x|) \sum_{m=-l}^l \alpha_{lm} Y_l^m(\theta_1, \phi_1). \end{aligned} \quad (3.6)$$

The series representation is valid in $B_r(0)$, with

$$\alpha_{lm} = ik \int_{\Gamma} h_l(k|y|) \varphi(y) Y_l^{-m}(\theta_y, \phi_y) dS(y). \quad (3.7)$$

Once again, we would like to approximate $f(x_0)$ by the partial sum of this expansion:

$$f(x_0) \approx f^{(N)}(x_0) = \sum_{l=0}^N j_l(k|x_0|) \sum_{m=-l}^l \alpha_{lm} Y_l^m(\theta_0, \phi_0). \quad (3.8)$$

To estimate the error $|f(x_0) - f^{(N)}(x_0)|$, we use an argument very much like that used to obtain (2.50) and (2.51). Before we turn to that task, however, let us compute the series expansion of the double layer

$$v(x) = \int_{\Gamma} \partial_{\mathbf{n}_y} G_k(x, y) \varphi(y) dS(y). \quad (3.9)$$

$v(x)$ also has an expansion like that in the second line of (3.6),

$$v(x) = \sum_{l=0}^{\infty} j_l(k|x|) \sum_{m=-l}^l \alpha_{lm} Y_l^m(\theta_1, \phi_1), \text{ for } |x| < r, \quad (3.10)$$

where the coefficients are now defined by the integrals:

$$\alpha_{lm} = ik \int_{\Gamma} \partial_{\mathbf{n}_y} [h_l(k|y|) Y_l^{-m}(\theta_y, \phi_y)] \varphi(y) dS(y). \quad (3.11)$$

The value of $v(x_0)$ can again be approximated by taking a partial sum of the series in (3.10).

The error estimates for both of these series approximations come from determining the r dependence of the coefficients in these expansion. We carry out the analysis for the single layer and leave the details for the double layer to the interested reader.

The necessary estimates follow from integration by parts using the fact that:

$$L_{\pm} Y_l^m = \sqrt{(l \mp m)(l \pm m + 1)} Y_l^{m \pm 1}, \quad (3.12)$$

where the vector fields are defined in (θ, ϕ) -coordinates by:

$$L_{\pm} = e^{\pm i\theta} [\partial_{\phi} \pm i \cot \phi \partial_{\theta}]. \quad (3.13)$$

An elementary calculation shows that, as operators on $L^2(S^2)$, the adjoints of L_\pm are $-L_\pm$. In integrating by parts, we need to apply these vector fields to $g(r\omega(\theta, \phi))$, where g is a function smooth in the closure of the sphere of radius r . An elementary calculation gives:

$$L_\pm g(r\omega(\theta, \phi)) = r \cos \phi [g_x \pm i g_y](r\omega(\theta, \phi)) - r e^{\pm i\theta} \sin \phi g_z(r\omega(\theta, \phi)). \quad (3.14)$$

It is useful to observe that:

$$\cos \phi = \sqrt{\frac{4\pi}{3}} Y_1^0 \text{ and } e^{\pm i\theta} \sin \phi = \sqrt{\frac{8\pi}{3}} Y_1^{\pm 1}, \quad (3.15)$$

and therefore

$$L_\pm g(r\omega(\theta, \phi)) = r \sqrt{\frac{4\pi}{3}} [Y_1^0 (g_x \pm i g_y) - \sqrt{2} Y_1^{\pm 1} g_z](r\omega(\theta, \phi)). \quad (3.16)$$

The Clebsch-Gordan relations [23] (or [22], 34.3.20, 34.2.1, 34.2.3), show that there are collections of coefficients $\{c_{jlm;n}^+; c_{jlm;n}^-\}$ so that for $j \in \{-1, 0, 1\}$ we have the identities

$$Y_1^j Y_l^{-m} = \sum_{n=1-l}^{l-1} c_{jlm;n}^- Y_{l-1}^n + \sum_{n=-(1+l)}^{1+l} c_{jlm;n}^+ Y_{l+1}^n. \quad (3.17)$$

An elementary computation shows that

$$|Y_1^j Y_l^{-m}(\omega)|^2 \leq \frac{3}{4\pi(1+|j|)} |Y_l^{-m}(\omega)|^2. \quad (3.18)$$

Integrating this inequality over the sphere, and using the orthogonality relations satisfied by the functions $\{Y_l^m\}$, we easily deduce that for $j \in \{-1, 0, 1\}$,

$$\sum_{n=1-l}^{l-1} |c_{jlm;n}^-|^2 + \sum_{n=-(l+1)}^{l+1} |c_{jlm;n}^+|^2 \leq \frac{3}{4\pi(1+|j|)}. \quad (3.19)$$

As before, we observe that the representation $f(r\omega(\theta, \phi))$ of the single layer potential in (3.6) is unique, and therefore the coefficients can be computed by integration on a sphere of radius r about the origin:

$$\tilde{\alpha}_{lm}(r) = j_l(kr) \alpha_{lm} = \int_0^{2\pi} \int_0^\pi f(r\omega(\theta, \phi)) Y_l^{-m}(\theta, \phi) \sin \phi d\phi d\theta. \quad (3.20)$$

For simplicity assume that $m \geq 0$, and use (3.12) to see that

$$L_+ Y_l^{-m} = \sqrt{(l+m)(l-m+1)} Y_l^{1-m}. \quad (3.21)$$

Integrating by parts in the integral defining $\tilde{\alpha}_{lm}(r)$ we see that

$$\tilde{\alpha}_{lm}(r) = - \int_0^{2\pi} \int_0^\pi L_+[f(r\omega(\theta, \phi))] \frac{Y_l^{1-m}(\theta, \phi)}{\sqrt{(l+m)(l-m+1)}} dS(\theta, \phi) \quad (3.22)$$

Using the relations in (3.16) and (3.17), we see that the integrand becomes

$$\begin{aligned}
& r \sqrt{\frac{4\pi}{3}} \frac{[(f_x + if_y)Y_1^0 - \sqrt{2}f_z Y_1^1]Y_l^{1-m}}{\sqrt{(l+m)(l-m+1)}} = \\
& r \sqrt{\frac{4\pi}{3(l+m)(l-m+1)}} \left\{ (f_x + if_y) \left[\sum_{n=1-l}^{l-1} c_{0lm;n}^- Y_{l-1}^n + \sum_{n=-(1+l)}^{1+l} c_{0lm;n}^+ Y_{l+1}^n \right] - \right. \\
& \quad \left. \sqrt{2}f_z \left[\sum_{n=1-l}^{l-1} c_{1lm;n}^- Y_{l-1}^n + \sum_{n=-(1+l)}^{1+l} c_{1lm;n}^+ Y_{l+1}^n \right] \right\}. \quad (3.23)
\end{aligned}$$

The formula in (3.23) has several notable features:

1. The entire quantity is multiplied by r .
2. The terms in these sums are of exactly the same types as those that appear in the original definition, with the integrals computing spherical harmonic expansion coefficients of derivatives of f along $|x| = r$.
3. Starting with a spherical harmonic of degree l we obtain terms of degree $l-1$ and $l+1$, showing that this procedure can be repeated l times.
4. The coefficients $\{c_{jlm;n}^\pm\}$ can be bounded by the identity in (3.19).

Using these observations, we carry out integration by parts $N+1$ times to obtain an estimate for the remainder:

$$e^{(N)}(x_0) = f(x_0) - f^{(N)}(x_0) = \sum_{l=N+1}^{\infty} \sum_{m=-l}^l \tilde{\alpha}_{lm}(r) Y_l^m(\theta_0, \phi_0). \quad (3.24)$$

Integrating the right hand side of (3.23) over the unit sphere leads to the computation of certain spherical harmonic coefficients of $(f_x + if_y)$, and $\sqrt{2}f_z$ restricted to the sphere of radius r . In order to estimate the coefficients $\{\tilde{\alpha}_{lm}(r)\}$, we need to introduce some notation. We let

$$D_1 = \partial_x + i\partial_y, \quad D_2 = \partial_x - i\partial_y, \quad D_3 = \sqrt{2}\partial_z. \quad (3.25)$$

For a 3-multi-index $\gamma = (\gamma_1, \gamma_2, \gamma_3)$, we let $\{\hat{\alpha}_{lm}^{(\gamma)}(r)\}$ denote the spherical harmonic coefficients of $D^\gamma f|_{bB_r(0)}$, where D^γ is the constant coefficient differential operator $D_1^{\gamma_1} D_2^{\gamma_2} D_3^{\gamma_3}$, so that

$$\hat{\alpha}_{lm}^{(\gamma)}(r) = \int_0^{2\pi} \int_0^\pi [D^\gamma f](r\omega(\theta, \phi)) Y_l^m(\theta, \phi) \sin \phi d\phi d\theta. \quad (3.26)$$

To obtain estimates for $\{\tilde{\alpha}_{lm}(r)\}$ that are proportional to r^{N+1} , we need to integrate by parts in (3.20) $N+1$ times. Each integration by parts increases the number of terms by a factor of 4. Whether one chooses L_+ or L_- at each step is not too important, but for the fact that for any pair (l, m) one can always choose the sign so that coefficient on the right hand side of (3.12) does not vanish.

Our formula expresses

$$\sum_{m=-l}^l \tilde{\alpha}_{lm}(r) Y_l^m(\theta, \phi) \quad (3.27)$$

as a sum of $(2l+1) \times 4^{N+1}$ sums, each consisting of terms of the form

$$\left(\frac{4\pi}{3}\right)^{\frac{N+1}{2}} r^{N+1} \cdot \widehat{\alpha}_{l_1 v_1}^{(\gamma)}(r) C_1 \cdots C_{N+1} Y_l^{u_{N+1}}(\theta, \phi). \quad (3.28)$$

Each matrix C_q is of the form

$$C_q = \frac{c_{j_q l_q u_q; v_q}^{\pm}}{d_q}. \quad (3.29)$$

For each q , the pair u_q, v_q are the matrix indices in C_q ,

$$j_q \in \{-1, 0, 1\}, \quad (3.30)$$

$$l - (N+1) \leq l_q \leq l + (N+1), \text{ and therefore} \quad (3.31)$$

$$u_q \in \{-l_q, \dots, l_q\} \text{ and } v_q \in \{-(l_q \pm 1), \dots, (l_q \pm 1)\}. \quad (3.32)$$

We think of $\widehat{\alpha}_{l_1 v_1}^{(\gamma)}(r)$ as a $1 \times [2(l_1 \pm 1) + 1]$ vector (indexed by v_1), and $Y_l^{u_{N+1}}$ as a $(2l+1) \times 1$ vector of functions (indexed by u_{N+1}). Moreover γ is a 3-multi-index satisfying $|\gamma| = N+1$.

Finally, the denominator is of the form

$$d_q = \sqrt{(l_q \mp u_q)(l_q \pm u_q + 1)}, \quad (3.33)$$

with $+$ or $-$ chosen so that d_q never vanishes. With this understood, d_q is easily seen to satisfy the estimate:

$$\sqrt{2l_q} \leq d_q. \quad (3.34)$$

Using (3.19) and (3.34), we can bound the Frobenius norm of each matrix C_q , in (3.28) (with u_q and v_q as the matrix indices) by $\sqrt{3/4\pi}$. Hence the absolute value of the quantity in (3.28) is bounded by

$$r^{N+1} \|\widehat{\alpha}_{l_1}^{(\gamma)}(r)\| \sqrt{\frac{2l+1}{4\pi}}, \quad (3.35)$$

where we use the fact that pointwise

$$\sum_{m=-l}^l |Y_l^m(\theta, \phi)|^2 = \frac{2l+1}{4\pi}. \quad (3.36)$$

It therefore follows that for some pairs $\{(\gamma_j, p_j) : j = 1, \dots, (2l+1) \cdot 4^{N+1}\}$ where $|\gamma_j| = N+1$ and $p_j \in \{-(N+1), \dots, (N+1)\}$, chosen as described above, we have

$$\left| \sum_{m=-l}^l \widetilde{\alpha}_{lm}(r) Y_l^m(\theta, \phi) \right| \leq \sqrt{\frac{2l+1}{4\pi}} r^{N+1} \sum_{j=1}^{(2l+1) \cdot 4^{N+1}} \|\widehat{\alpha}_{(l-p_j)}^{(\gamma_j)}\|. \quad (3.37)$$

Finally, collecting terms we see that there is a constant K_N so that:

$$|e^{(N)}(x_0)| \leq K_N r^{N+1} \sum_{\gamma \in J_{3, N+1}} \sum_{l=0}^{\infty} \left[\sum_{m=-l}^l |\widehat{\alpha}_{lm}^{(\gamma)}(r)|^2 \right]^{\frac{1}{2}} (l+N+1)^{\frac{3}{2}} \quad (3.38)$$

Here $J_{3,N+1}$ is the collection of 3-multi-indices with $|\gamma| = N + 1$. In order for these infinite sums to be finite, we need to assume that, for each $\gamma \in J_{3,N+1}$ the sum satisfies:

$$\sum_{m=-l}^l |\hat{\alpha}_{lm}^{(\gamma)}(r)|^2 = O\left(\frac{1}{l^{5+\delta}}\right), \quad (3.39)$$

for some $\delta > 0$.

The estimate in (3.39) holds if, for each $\gamma \in J_{3,N+1}$, the function $D^\gamma f \upharpoonright_{bB_r(0)}$ belongs to the L^2 -Sobolev space $W^{3+\delta,2}(bB_r(0))$. For a single layer, this would follow from assuming that $\varphi \in W^{3+N+\delta,2}(\Gamma)$. This represents a small loss of regularity over the estimates we obtained in two dimensions. This loss would appear to be a result of the added complexity of multiplying spherical harmonics rather than exponentials. A more careful estimate of the terms in (3.28) might provide a somewhat better result.

THEOREM 3.1. *Let k be a complex number with $\text{Im } k \geq 0$ and let Γ denote a smooth hypersurface in \mathbb{R}^3 , such that $B_r(0) \subset \Gamma^c$ with $x_0 = r\omega(\theta_0, \phi_0) \in \Gamma \cap bB_r(0)$. For each positive integer N and $\delta > 0$, there is a constant $M_{N,\delta}$ such that*

$$\left| \int_{\Gamma} G_k(x_0, y) \varphi(y) dS(y) - \sum_{l=0}^N j_l(kr) \sum_{m=-l}^l \alpha_{lm} Y_l^m(\theta_0, \phi_0) \right| \leq M_N r^{N+1} \|\varphi\|_{W^{3+N+\delta,2}(\Gamma)}, \quad (3.40)$$

so long as $\varphi \in W^{3+N+\delta,2}(\Gamma)$. The coefficients $\{\alpha_{lm}\}$ are given by (3.7).

4. Quadrature error. The prior estimates establish bounds on the tails of the expansions used by QBX, and hence on the error incurred in their truncation. They do not take into account that the coefficients in these expansions must still be computed numerically. In this section, we derive an error estimate for this quadrature problem. This estimate then helps to establish an upper bound on the “mesh spacing” h needed, leading to straightforward control of the overall error in QBX, which is bounded above by the sum of these truncation and quadrature errors.

Concentrating for the moment on the single layer for the Laplace equation in two dimensions, let us first rewrite (2.26) as

$$e_N(\xi_1, \xi_2) = u(\xi_1, \xi_2) - A_0 - \text{Re} \left\{ \sum_{j=1}^N a_j z^j \right\} = \text{Re} \left\{ \sum_{j=N+1}^{\infty} a_j z^j \right\} \quad (4.1)$$

where

$$A_0 = \frac{1}{2\pi} \int_0^L \varphi(w(t)) \log |w(t)| dt,$$

$$a_j = \frac{1}{2\pi} \int_0^L \frac{\varphi(w(t))}{j[w(t)]^j} dt,$$

and $\Gamma = w([0, L])$. Recall, also, that the center z_c here is assumed to be the origin, for simplicity of notation. In practice, what we wish to estimate is actually

$$e_N(\xi_1, \xi_2) = \left| u(\xi_1, \xi_2) - A_0^Q - \text{Re} \left\{ \sum_{j=1}^N a_j^Q z^j \right\} \right|, \quad (4.2)$$

where A_0^Q and a_j^Q are the *computed approximants* of A_0 and a_j . We write this error as

$$e_N(\xi_1, \xi_2) = \left| \left(u(\xi_1, \xi_2) - A_0 - \operatorname{Re} \left\{ \sum_{j=1}^N a_j z^j \right\} \right) - (A_0^Q - A_0) - \operatorname{Re} \left\{ \sum_{j=1}^N [a_j^Q - a_j] z^j \right\} \right|. \quad (4.3)$$

The first term is $O(r^{N+1})$ as shown in Theorem 2.2. The second term depends strongly on the specific quadrature rule used. To facilitate straightforward generalization to complicated geometries in more than two dimensions, we choose a composite Gauss quadrature for this role. If the boundary Γ were divided into M equal subintervals of size h , and a q th order Gauss quadrature rule were used on each subinterval, then

$$\left| (A_0^Q - A_0) - \operatorname{Re} \left\{ \sum_{j=1}^N [a_j^Q - a_j] z^j \right\} \right| = C_q(N, \Gamma) \left(\frac{h}{4r} \right)^{2q} \|\varphi\|_{\mathcal{C}^{2q}}. \quad (4.4)$$

To see this, we note that on a curve segment Γ_i of length h , the standard estimate for q -point Gauss-Legendre quadrature [8] yields

$$\left| \int_{\Gamma_i} \frac{\varphi(w(t))}{j[w(t)]^j} dt - \sum_{n=1}^q \frac{\varphi(w(t_n))}{j[w(t_n)]^j} w_n \right| \leq \frac{h^{2q+1}}{j(2q+1)} \frac{(q!)^4}{(2q)!^3} \left| D^{2q} \frac{\varphi(w(t))}{[w(t)]^j} \right|_{\infty}, \quad (4.5)$$

where D^q denotes the q^{th} derivative of the integrand with respect to the integration parameter along Γ_i , and $\{t_n, w_n\}$ denote the Gauss nodes and weights scaled to Γ_i . From Stirling's approximation

$$\sqrt{2\pi n}^{n+\frac{1}{2}} e^{-n} < n! < 2\sqrt{\pi n}^{n+\frac{1}{2}} e^{-n},$$

we have

$$\left| \int_{\Gamma_i} \frac{\varphi(w(t))}{j[w(t)]^j} dt - \sum_{n=1}^q \frac{\varphi(w(t_n))}{j[w(t_n)]^j} w_n \right| \leq \frac{h^{2q+1}}{(2q)! 4^{2q}} \left| D^{2q} \frac{\varphi(w(t))}{[w(t)]^j} \right|_{\infty}.$$

Omitting a detailed computation of the derivative, we note that it satisfies a bound of the general form

$$\left| D^{2q} \frac{\varphi(w(t))}{[w(t)]^j} \right|_{\infty} \leq C_q(N, \Gamma) \frac{(2q)!}{r^{j+2q}} \|\varphi\|_{\mathcal{C}^{2q}}.$$

The constant in this estimate depends on the order of the local expansion N and on the smoothness of the curve Γ , through a polynomial in $(w, w^{(1)}, \dots, w^{(2q)})$. Since the dependence of the constant on N , and the parametrization are rather complicated, but independent of φ , we leave it in the form above.

Adding up the errors from all subintervals yields (4.4). The total error incurred by the QBX method can therefore be written in the form

$$E = O(r^{N+1}) + C_q(N, \Gamma) \left(\frac{h}{4r} \right)^{2q} \|\varphi\|_{C^{2q}}.$$

Fixing the order of the expansion N and letting $r = 4h$ yields an error of the form:

$$E = O(h^{N+1}) + C_q(N, \Gamma) \left(\frac{1}{16} \right)^{2q} \|\varphi\|_{C^{2q}}.$$

Implemented in this manner, the QBX method is asymptotic, converging like a method of order $N+1$, until the error is dominated by the second term. To obtain a *convergent* scheme one can, for example, set $r = \sqrt{h}$ so that

$$E = O(h^{\frac{N+1}{2}}) + C_q(N, \Gamma) \left(\frac{h}{16} \right)^q \|\varphi\|_{C^{2q}}.$$

Similar estimates can be obtained for Helmholtz layer potentials and for three-dimensional cases. ■

In practice, when dealing with complicated boundaries, adaptive discretization is generally required, and the assumption that the boundary is divided into equal subintervals must be relaxed. This does not change the error analysis in a substantial way, and we refer the reader to [16] for details and examples.

5. Conclusions. It is well-known that local expansions of layer potentials are analytic away from the boundary. There are surprisingly few results in the literature, however, concerning the behavior of such expansions when evaluated at the limit of their radii of convergence - namely at the nearest boundary point itself. The estimates in Theorems 2.1, 2.2, 2.4, 2.5, and 3.1 show that, despite the absence of a geometric decay parameter, high order accuracy can be obtained in computing layer potentials in a manner that is controlled by the smoothness of the boundary data. The analysis presented here serves as a complement to [16], providing a convergence theory for the quadrature method QBX.

We note that sharper results can be obtained [3], if some assumptions are made about analyticity of the data and the location of the nearest singularity.

Finally, estimates of the type obtained here may be of interest in areas of mathematics that involve smooth continuation, microlocal analysis, and potential theory.

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